Introduction to Mathematics and Modeling

lecture 1

Functions and Trigonometry

UNIVERSITY OF TWENTE.

academic year	:	18-19
lecture	:	1
build	:	November 13, 2018
slides	:	25

This week

intro

GELGOOG Coffee Beans Husking Machine

1 Section 1.1: functions and their graphs
2 Section 1.2: combining and transforming functions
3 Section 1.3: trigonometry

About this course

- Monday : lecture LEC
  Tuesday/Thursday: assisted self-tuition AST
- Assisted self-tuition:

  Basic exercises : Ⓑ
  Advanced exercises: [A]

- Three midterm tests and one resit. See MyTimeTable for date, time and location.
- Test 1: hand written test,
  Test 2: hybrid test (hand written test and MyLabsPlus test),
  Test 3: MyLabsPlus test.
- Examples and some exercises with *Mathematica*.
- Course schedule, slides and other materials can be found via the link on the Canvas page **Smart Environments (2018-1B)**, module **ItE: Introduction to Mathematics and Modeling I**, page **Lecture slides**.

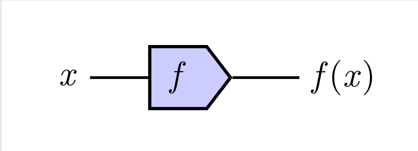
Topics of this course

Nr	Week	Topic
1	1	Basics: functions, graphs and trigonometry
2	2	Basics: the inverse; exponential functions and logarithms
		Midterm test 1
3	3	Differentiation: definition
4	4	Differentiation: rules and properties
5	5	Differentiation: applications
		Midterm test 2
6	6	Integration: definition and applications
7	7	Integration: the fundamental theorem; method of substitution
8	8	Integration: integration by parts
		Midterm test 3
		Resit

Definition

A **function** $f\colon D \to C$ *is a rule that assigns a unique element* $f(x)$ *in* C *to each element* x *in* D*.*

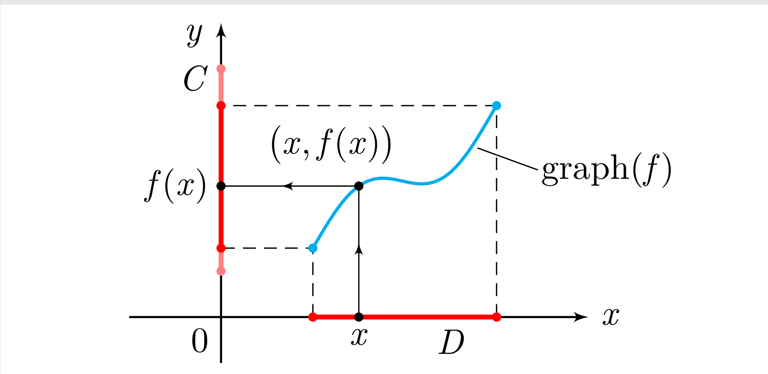
- The set D is the **domain** of f.
- The set C is the **codomain** of f.
- The **range** or **image** of f is the set of all function values $f(x)$.
- If f assigns y to x, then we denote this as $y = f(x)$ or $x \mapsto f(x)$.
- In a **diagram**:



Definition

Let D *and* C *are subsets of* $\mathbb{R}$*. The* **graph** *of a function* $f\colon D \to C$ *is defined as*

$$\operatorname{graph}(f) = \{(x, f(x)) \mid x \in D\}.$$



Vertical Line Test

A vertical line intersects the graph of a function in at most one point.

Mathematica

- *Defining a function:*

```
f[x_]:=1/Sqrt[x^2+1]
```

- *Plotting a function f with domain $[a, b]$:*

```
Plot[f[x],{x,a,b}]
```

Empty notebook (`Worksheet 1.nb`)

Definition

Let the function f be definied by a formula.

- *If the domain of f is not defined explicitly, then the domain consists of all numbers x for which $f(x)$ exists.*
- *If the codomain of f is not defined explicitly, then the codomain is chosen as large as possible.*

Example:

Let $f(x) = \sqrt{x-3}$.

- The expression $\sqrt{x-3}$ is defined for all x for which $x - 3 \geq 0$, hence

$$\mathrm{Dom}(f) = [3, \infty).$$

- The codomain is $\mathbb{R}$.
- The range is $[0, \infty)$.

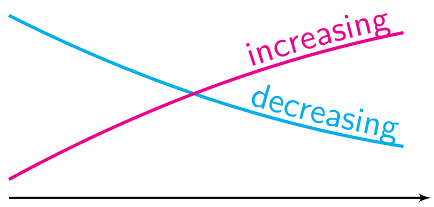
Monotony

Definition

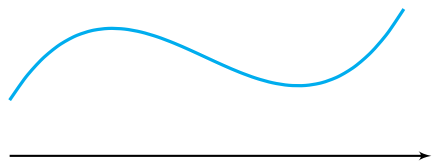
Let $f\colon I \to \mathbb{R}$ be a function defined on an interval I.

1. *The function f is* **increasing on** I *if for all $x_1, x_2 \in I$ with $x_1 < x_2$:*
   $$f(x_1) < f(x_2).$$
2. *The function f is* **decreasing on** I *if for all $x_1, x_2 \in I$ with $x_1 < x_2$:*
   $$f(x_1) > f(x_2).$$

- A function that is decreasing or increasing is called **monotonous**.



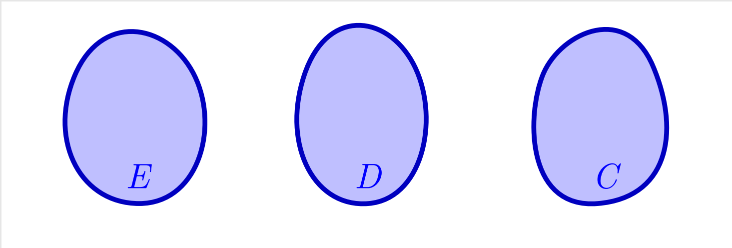
monotonous


not monotonous

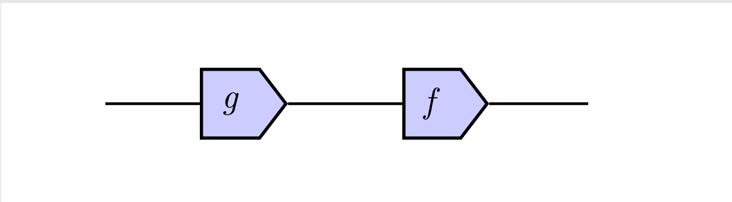
Algebraic combinations

Addition:	$h(x) = f(x) + g(x)$	$f + g$
Subtraction:	$h(x) = f(x) - g(x)$	$f - g$
Multiplication:	$h(x) = f(x)g(x)$	$f\,g$
Division:	$h(x) = \dfrac{f(x)}{g(x)}$	$\dfrac{f}{g}$
Composition:	$h(x) = g(f(x))$	$g \circ f$

Composition 2.2



- Let $f\colon D \to C$ and $g\colon E \to D$ be two functions. The domain of f contains the range of g.
- The **composition of** f **and** g is defined as the function $f \circ g\colon E \to C$ that assigns the element $f(g(x))$ to every $x \in E$.
- Pronounce $f \circ g$ as "f after g".

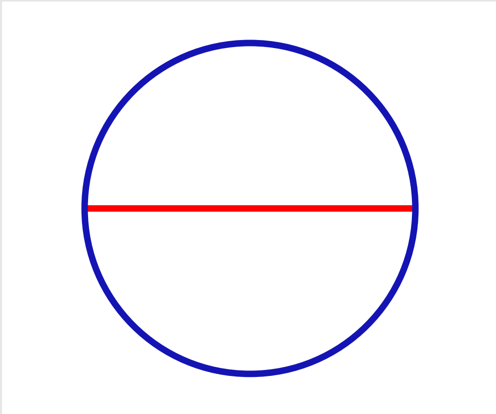


Exercises 2.3

(1) Define $f(x) = \sqrt{x}$ and $g(x) = x + 1$. Find $f \circ g$,and $g \circ f$.

(2) Define $f(x) = \sqrt{x - 1}$ and $g(x) = x^2 + 2$. Find the domains and the ranges of f, g, $f \circ g$ and $g \circ f$.

The number π

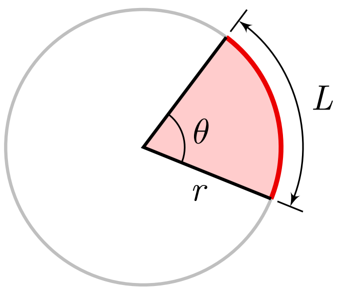


$$\pi = \frac{\text{circumference}}{\text{diameter}}$$

$$\pi \approx 3.14159265358979323846264338327950288419716939937510582\,0\ldots$$

Radians

Theorem

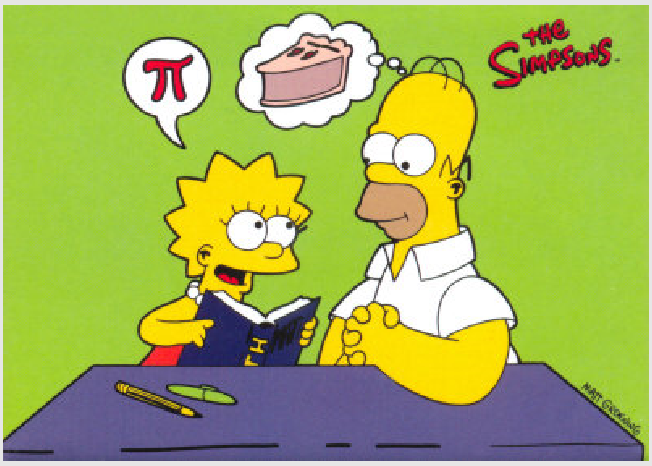


In a sector, the length of the arc is equal to the product of the angle of the sector (measured in radians) and the radius of the circle.

$$\boxed{L = r\theta}$$

- The **radian** is a unit for measuring angles.
- One radian is approximately 57.3 degrees.
- A full circle is 2π radians.

Pie

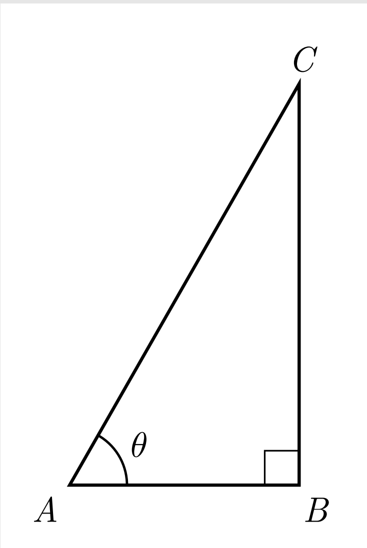


The Simpsons, (TM) Twentieth Century Fox

π = 

Sine, cosine and tangent for acute angles

Triangle ABC is rectangular ($\angle ABC = \frac{\pi}{2}$), angle at A is acute ($0 < \theta < \frac{\pi}{2}$).



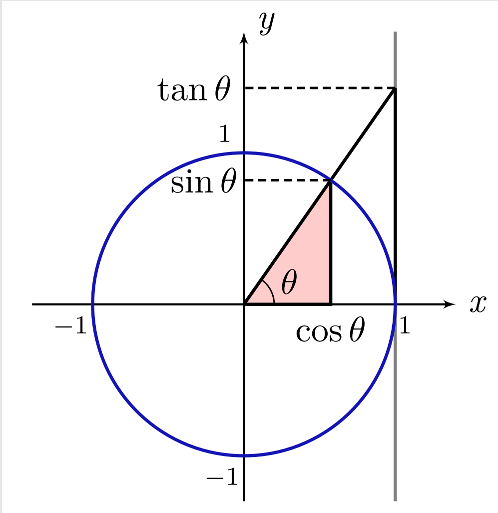
$$\cos\theta = \frac{AB}{AC},$$

$$\sin\theta = \frac{BC}{AC},$$

$$\tan\theta = \frac{BC}{AB} = \frac{\frac{BC}{AC}}{\frac{AB}{AC}} = \frac{\sin\theta}{\cos\theta}.$$

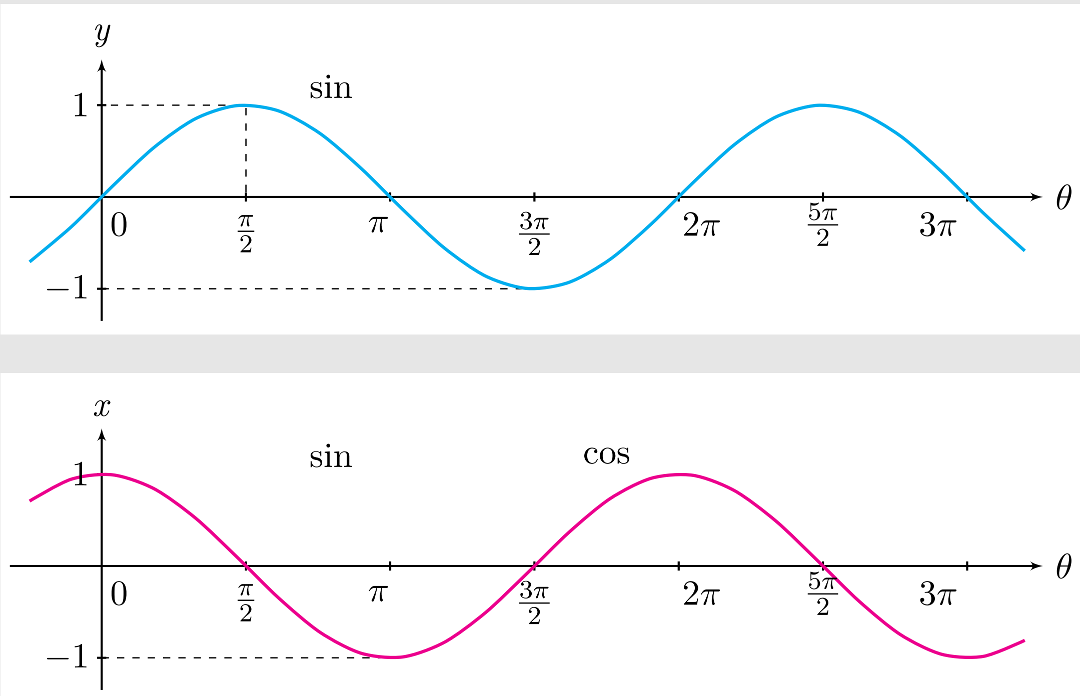
Sine and cosine for arbitrary angles 3.5

For arbitrary angles, the sine, cosine are defined with the **unit circle**: the circle with center $(0, 0)$ and radius 1.

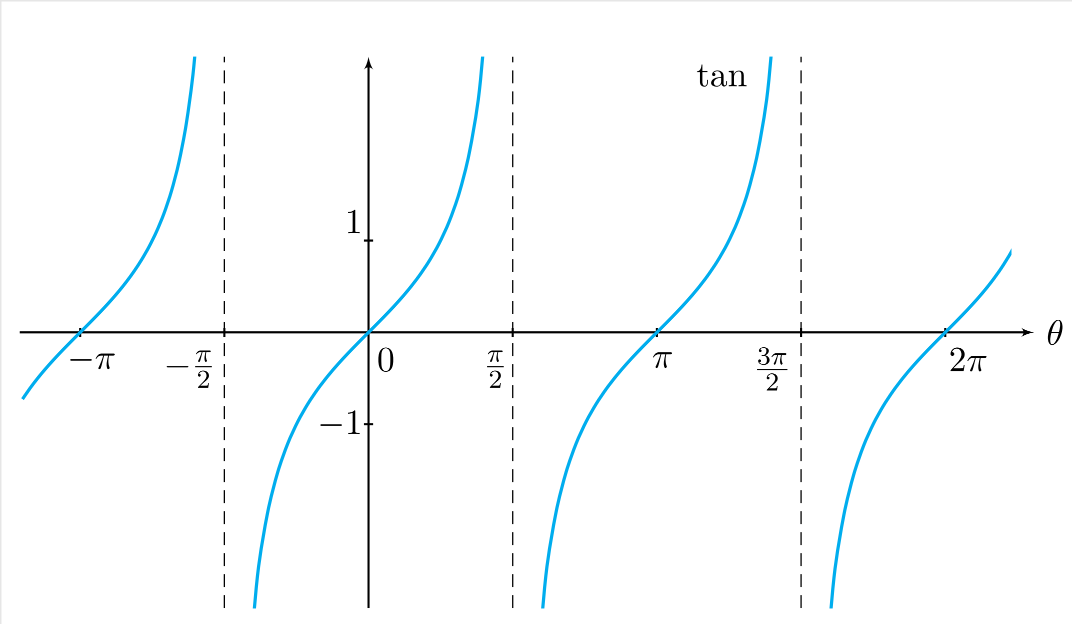


Trigonometry - SinCosTan.nb

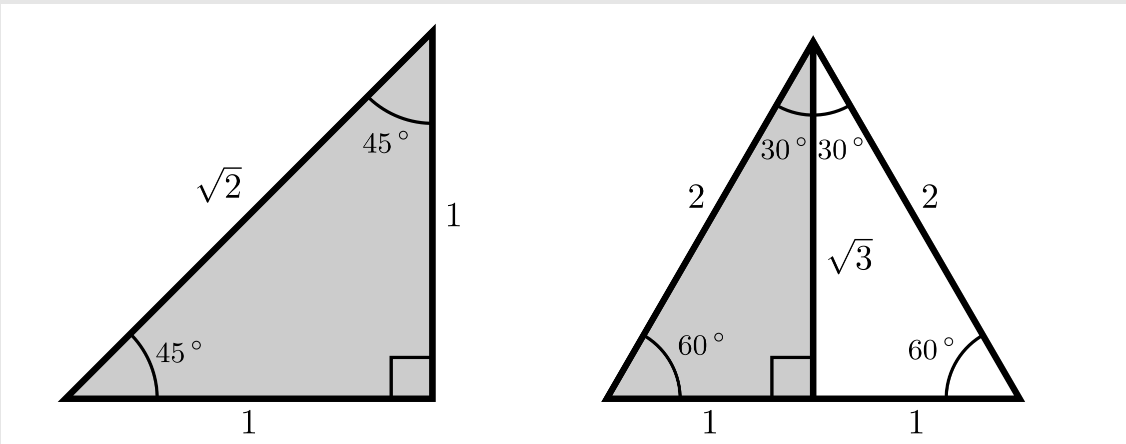
Graphs of sine and cosine 3.6



Graph of the tangent



Sine, cosine and tangent of special angles



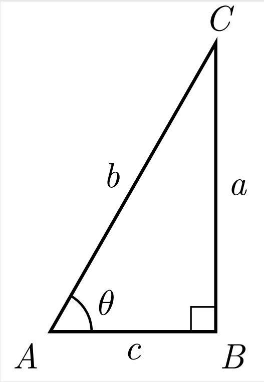
$$\cos\left(\frac{\pi}{4}\right) = \tfrac{1}{2}\sqrt{2} \qquad \cos\left(\frac{\pi}{6}\right) = \tfrac{1}{2}\sqrt{3} \qquad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\sin\left(\frac{\pi}{4}\right) = \tfrac{1}{2}\sqrt{2} \qquad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \qquad \sin\left(\frac{\pi}{3}\right) = \tfrac{1}{2}\sqrt{3}$$

$$\tan\left(\frac{\pi}{4}\right) = 1 \qquad \tan\left(\frac{\pi}{6}\right) = \tfrac{1}{3}\sqrt{3} \qquad \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

Pythagoras' theorem

4.1



$\cos^2\theta + \sin^2\theta =$

Sum rule for sine and cosine

4.2

Theorem

For arbitrary $\alpha, \beta \in \mathbb{R}$ *we have*

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

- **Example**: use $\frac{5}{12}\pi = \dfrac{\pi}{6} + \dfrac{\pi}{4}$ to calculate $\sin\left(\frac{5}{12}\pi\right)$.

Difference rule for sine and cosine

Theorem

For arbitrary $\alpha, \beta \in \mathbb{R}$ *we have*

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

- We prove the first equation:

$$\sin(\alpha - \beta) = \sin\left(\alpha + (-\beta)\right)$$
$$= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$
$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

Doubling angles

Theorem

For arbitrary $\alpha \in \mathbb{R}$ *we have*

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha,$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha.$$

Overview

Periodicity	$\sin(\alpha+2\pi)=\sin\alpha$ and $\sin(\alpha+\pi)=-\sin\alpha$ $\cos(\alpha+2\pi)=\cos\alpha$ and $\cos(\alpha+\pi)=-\cos\alpha$
Symmetry	$\sin(-\alpha)=-\sin\alpha$ $\cos(-\alpha)=\cos\alpha$
Congruence	$\sin\left(\alpha+\frac{\pi}{2}\right)=\cos\alpha$ and $\sin\left(\alpha-\frac{\pi}{2}\right)=-\cos\alpha$ $\cos\left(\alpha+\frac{\pi}{2}\right)=-\sin\alpha$ and $\cos\left(\alpha-\frac{\pi}{2}\right)=\sin\alpha$
Sum formulas	$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$ $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
Difference formulas	$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$ $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$
Doubling formulas	$\sin(2\alpha)=2\sin\alpha\cos\alpha$ $\quad\sin^2\left(\frac{1}{2}\alpha\right)=\frac{1}{2}-\frac{1}{2}\cos\alpha$ $\cos(2\alpha)=\cos^2\alpha-\sin^2\alpha$ $\quad\cos^2\left(\frac{1}{2}\alpha\right)=\frac{1}{2}+\frac{1}{2}\cos\alpha$
Pythagoras' thm	$\cos^2\alpha+\sin^2\alpha=1$

Exercises

(1) a) Use $\dfrac{\pi}{12}=\dfrac{\pi}{3}-\dfrac{\pi}{4}$ to calculate $\cos\left(\dfrac{\pi}{12}\right)$.

b) Calculate $\sin\left(\dfrac{\pi}{12}\right)$.

c) Using the doubling formulas, check the results of a) and b).

(2) Prove the doubling formulas for sine and cosine.